Mathematical Proofs

And how they are applied to Computer Olympiads

Counter Example

Can ONLY be used to DISPROVE a proposition, NEVER to PROVE it, by not being able to find one!

You have to prove there is no counter example, for it to prove the proposition.

Proof by contradiction

Basic idea:

You want to prove a proposition true.

- Assume that the given proposition is untrue.
- Based on that assumption reach a statement that is impossible.
- Thus the assumption is false.
- Hence the proposition must be true.

Example 1

Prove there is no largest prime, i.e. there are an infinite amount of them.

Proof by contradiction:

- Assume there are not infinitely many primes
- Call them: p_1, p_2 , p_3, $p_4 \ldots p_n$
- Consider the number $x = p_1 * p_2 * p_3 * p_4 \ldots * p_n + 1$
- x is clearly not prime because since it does not equal p_1, p_2 , p_3, $p_4 \ldots p_n$
- Known Fact: Any number has a prime devisor

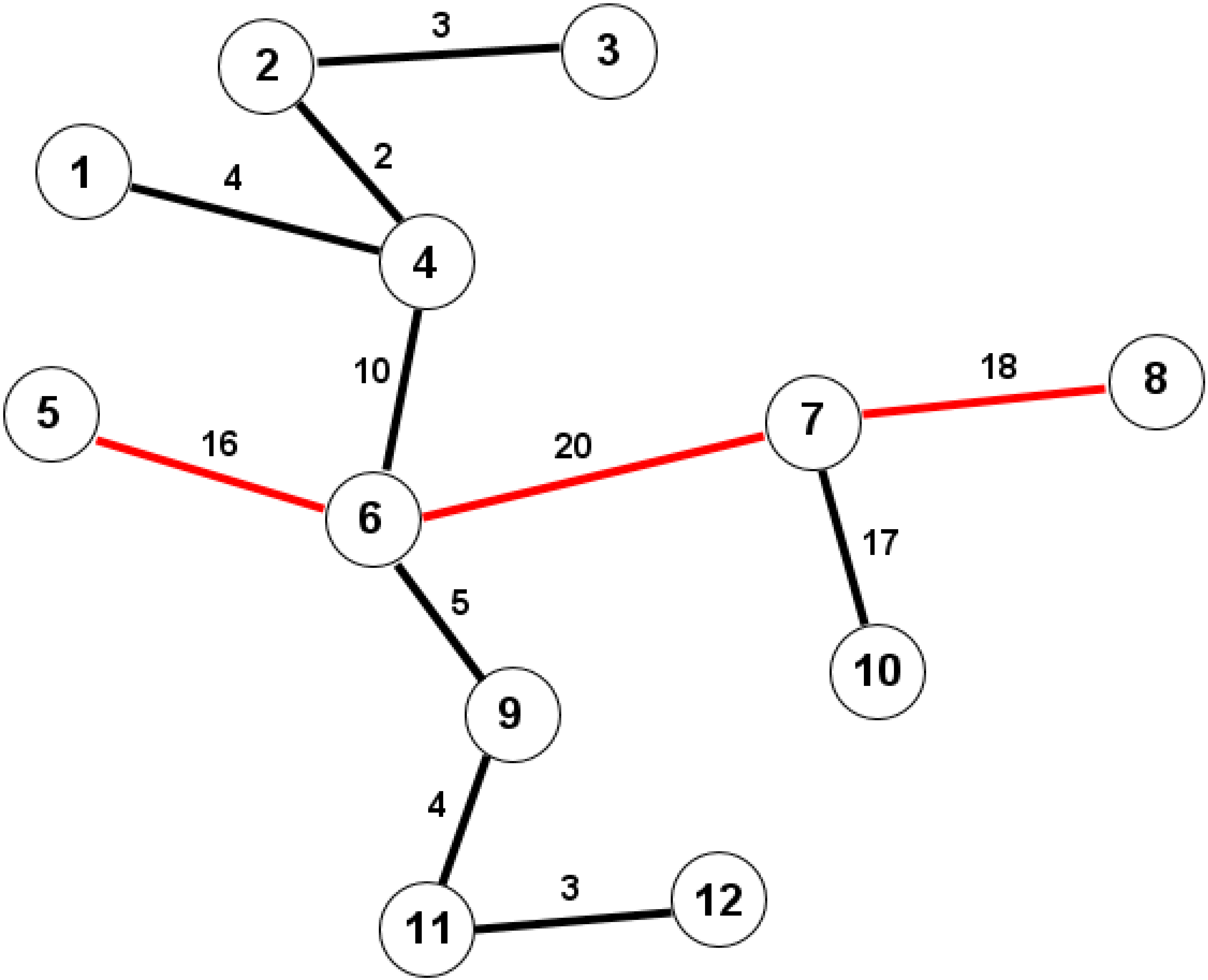
- Thus x has a prime devisor p_k
- But we have that p_1, p_2 , p_3, $p_4 \dots p_n$ do not divide x
- Thus we have another prime p_k not equal to p_1, p_2 , p_3, $p_4 \dots p_n$
- But we assumed p_1, p_2 , p_3, $p_4 \dots p_n$ are all the primes
- Thus a contradiction and we have proved there are infinitely many primes and hence no largest prime.

Example 2

Grand Central Taxi Rank (2008 Camp 2 day 2)

You are given a weighted spanning tree.

Find the node that minimises the maximum distance to any other node.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Example 3

Making Change (Online Camp 2008)

- You are given a coin system: c_1, c_2 , c_3, $c_4 \ldots c_n$
- There is a "Greedy" method to make any amount.
- Find if greedy algorithm is optimal i.e. do you use the minimal amount of coins.
- If not, give the smallest amount for witch it is not optimal.
- Small Example: coins 1, 2, 4, 5 and you want to make 8
- Greedy: 5+ 2 +1 = 8
- Optimal: 4 + 4 = 8
- Thus greedy is not optimal

Solution to Making Change

We need to find the smallest value for which the greedy solution is not optimal or prove that the greedy solution is always optimal.

Thus test all amounts up to a certain point x.

Greedy method:

For any given amount A, take the largest coin, C, that is smaller than that amount. Repeat for A = A-C until A=0.

This method is guaranteed to work if you have a coin with value 1.

DP Method:

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for i from 1 to x:            //x is the point where we stop
   testing     num[i] = 0

for i from 1 to x:
   for j from 0 to n-1:       //m is amount of coins
        if coin[j] > i:
              break
        num[i] = min(num[i], num[i — coins[j]] + 1)
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Suppose we have a coin system where a greedy solution is not optimal.

The smallest value for this coin system for which the greedy method is non optimal is x.

Assume $x > 2m$, where m is the biggest coin.

- minCoin[x] = k and greed[x] = g where k < g.
- We use a certain coin, call it d, in the optimal method to make x
- Consider amount x-d, we have minCoin[x-d] = k-1
- Since x > 2m we have x-d > m.
- Since x-d < x the greedy method for x-d is optimal.
- The greedy method to make value x-d will use coin m
- Since we obtain value x from (x-d) +d we have the optimal solution for x contains coin m
- Thus we use coin m in the optimal strategy to make x-d and we will use m in the optimal strategy to make value x.

Proof by contradiction continued

- Thus minCoin[x-m] = k-1
- The greedy solution for x also contains m.
- Thus greed[x-m] = g -1
- Then we have k-1 = g-1
- Thus contradiction

Thus x <= 2m